Bohr Almost Periodicity and Functions of Dynamical Type

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In this note we define the space $BAP(G)$ of Bohr almost periodic functions on a locally compact group G and, after reviewing the basic implications of the definition, discuss examples of functions that are Bohr almost periodic, but not almost periodic in the sense of Bochner. These examples are either due to or inspired by T.-S. Wu. We then consider dynamical properties of $BAP(G)$, showing among other things that $BAP(G) \subset MIN(G)$, the space of minimal functions on G. We also mention some pathologies; for example, $BAP(G)$ need not be a linear space. A concluding result, which we quote, is due to A. L. T. Paterson and may be thought of as a regularity property of $BAP(G)$. It asserts that $BAP(G)$ consists of left almost convergent functions.

A way to view one aspect of Harald Bohr's achievement with his theory of almost periodic functions is that he provided a characterization of the norm closed, linear span of the continuous periodic functions on **R**. It is clear that any attempt to characterize this space must overcome the apparent problem that even the sum of two periodic functions is usually not periodic, e.g., $x \to \sin x + \sin(\sqrt{2}x)$. Earlier attempts at some aspects of such a characterization had been made by Bohl [3] and Esclangon [6].

Bohr defined a continuous complex-valued function f on **R** to be almost periodic if: for any $\epsilon > 0$, there is a real number $L(\epsilon) > 0$ such that every interval of length $L(\epsilon)$ contains at least one translation number of f corresponding to ϵ. (See Bohr [4; pp. 31-2], for example. [5] has an extensive bibliography.) We write this in symbols:

for any $\epsilon > 0$, there is a real number $L_\epsilon > 0$ such that $[r, r + L_\epsilon] \cap \{s \mid |f(t+s) - f(t))| < \epsilon$ for all $t \in \mathbf{R}\} \neq \emptyset$ $(r \in \mathbf{R})$. The rationale for the term "almost periodic" is obvious; if the real number $L_\epsilon > 0$ exists for $\epsilon = 0$, then f is periodic. Also obvious is how to generalize the setting.

Definition 1. A continuous complex-valued function f on a locally compact group G is called *Bohr almost periodic* if:

for any $\epsilon > 0$ there is a compact $K_\epsilon \subset G$ such that

$$(rK_\epsilon) \cap \{s \in G \mid |f(ts) - f(t)| < \epsilon \text{ for all } t \in G\} \neq \emptyset \ (r \in G). \tag{1}$$

Let $BAP(G)$ denote the class of Bohr almost periodic functions on G.

Although the definition does not require $f \in BAP(G)$ to be bounded, it does require

$$\|R_s f - f\| := \sup_{t \in G} |(R_s f - f)(t)| = \sup_{t \in G} |f(ts) - f(t)| < \epsilon$$

for many $s \in G$. Also, since $\{s \mid \| R_s f - f\| < \epsilon\}$ is a symmetric set, (1) is equivalent to

$$K_\epsilon \{s \in G \mid \|R_s f - f\| < \epsilon\} = G, \tag{1'}$$

and to

$$\text{for each } t \in G, \text{ there is a } k \in K_\epsilon \text{ such that } \|R_t f - R_k f\| < \epsilon. \tag{1''}$$

Since a function in $BAP(S)$ must in fact be bounded, the formulation (1'') shows that the compact sets K_ϵ can always be chosen finite if and only if the *orbit* $R_G f := \{R_s f \mid s \in G\}$ is totally bounded, i.e., f is almost periodic in the sense of Bochner [2]. Thus, denoting by $AP(G)$ the class of Bochner almost periodic functions, we note that $BAP(G) = AP(G)$ if G is discrete.

2. Here are some facts about $BAP(G)$. Their demonstration can usually be modelled on proofs in Bohr [4]; see also [8, 1]. ([10] is a standard reference for topological groups.)

(a) The functions in $BAP(G)$ are bounded (as mentioned above) and right uniformly continuous. (We write $BAP(G) \subset \mathscr{U}_r(G)$; a function $f\colon G \to \mathbf{C}$ is *right uniformly continuous* if, for all $\epsilon > 0$, there is a neighbourhood V of the identity $e \in G$ such that $|f(s) - f(t)| < \epsilon$ whenever $st^{-1} \in V$.)

(b) $BAP(G)$ is norm closed in $\mathscr{U}_r(G)$ and *translation invariant* (i.e., $f \in BAP(G)$ and $s \in G$ imply $R_s f$, $L_s f \in BAP(G)$, where $L_s f(t) = f(st)$).

(c) $BAP(G) \cap \mathscr{U}_l(G) = AP(G)$. (Here $\mathscr{U}_l(G)$ is the analogously defined space of bounded functions that are *left* uniformly continuous.)

From (a) and (c) it follows that $BAP(G) = AP(G)$ if $\mathscr{U}_r(G) = \mathscr{U}_l(G)$ (for example, if G is abelian). The converse is an open question. A group to look at in this connection is the affine group of the line $\mathbf{R} \otimes \mathbf{R}^+$, for which we suspect $BAP = AP$, although $\mathscr{U}_r \neq \mathscr{U}_l$.

The definition of $BAP(G)$ uses right translates. We denote by $LBAP(G)$ the space defined analogously using left translates. It follows that $LBAP \subset \mathscr{U}_l$, that $LBAP \cap BAP = AP$, and that the equality $LBAP = BAP$ implies the identity of all three spaces, $LBAP = BAP = AP$.

Examples 3. The examples presented here of functions in $BAP \setminus AP$ are due to or inspired by Wu [17]. A more detailed treatment of them can be found in [12, 13, 14].

(i) On $G = \mathbf{C} \otimes \mathbf{T}$, the euclidean group of the plane with multiplication $(z', w')(z, w) = (z' + w'z, w'w)$, the function $f(z, w) = e^{i\mathrm{Re}(z/w)}$ is in $BAP \setminus \mathscr{U}_l$, as is readily verified. (Here Re indicates real part.)
(ii) On $G = (\mathbf{T} \times \mathbf{T}) \otimes \mathbf{Z}$ with multiplication $(w_1', w_2', n')(w_1, w_2, n) = (w_1' w_1 w_2^{n'}, w_2' w_2, n' + n)$, the function $f(w_1, w_2, n) = w_1$ satisfies $R_{(1,1,m)} f = f$ for all $m \in \mathbf{Z}$. Hence $f \in BAP(G)$, since we can choose $K_\epsilon = \mathbf{T} \times \mathbf{T} \times \{0\}$ for all $\epsilon > 0$ in Definition 1. However $f \notin \mathscr{U}_l$. (This is Wu's method [17] and works more generally: if $G = G_1 \otimes G_2$ is a semidirect product with G_1 compact, and if $F \in C(G_1)$, then $f(s, t) = F(s)$ defines an $f \in BAP(G)$.)
(iii) Let $G = \mathbf{T}^{\mathbf{T}} \otimes \mathbf{T}_d$, where $\mathbf{T}^{\mathbf{T}}$ is the compact group of all functions from $\mathbf{T}$ into $\mathbf{T}$ and $\mathbf{T}_d$ is the discrete circle group. The product in G is $(h', w')(h, w) = (h' R_{w'} h, w'w)$. Let $f(h, w) = h(1)$. Then $f \in BAP \setminus \mathscr{U}_l$. Further, define $g \in \mathbf{T}^{\mathbf{T}}$ by $g(-1) = -1$, $g(w) = 1$ otherwise. Then, by 2(b), $R_{(g,1)} f \in BAP$. However $f + R_{(g,1)} f \notin BAP$.

We now want to make a connection with topological dynamics. If $f \in \mathscr{U}_r(G)$, then the closure $X_f := \overline{R_G f}$ of the orbit $R_G f$ in the topology of pointwise convergence on G is compact in $\mathscr{U}_r(G)$ for that topology. The translation operators $R_t, t \in G$, leave X_f invariant and (R_G, X_f) is a flow. f is called *minimal, point distal* or *distal* if that flow is minimal, point distal with f as distal point, or distal, respectively. Specifically, an $f \in \mathscr{U}_r(G)$ is:

minimal if, whenever $h_1 = \lim_\alpha R_{s_\alpha} f$ (pointwise on G), there is a net $\{t_\beta\} \subset G$ such that $f = \lim_\beta R_{t_\beta} h$;

point distal if, whenever $h_1 = \lim_\alpha R_{s_\alpha} f$ and $\lim_\beta R_{t_\beta} h_1 = h' = \lim_\beta R_{t_\beta} f$, it follows necessarily that $h_1 = f$;

or

distal if, whenever $h_1 = \lim_\alpha R_{s_\alpha} f$, $h_2 = \lim_\beta R_{t_\beta} f$ and $\lim_\gamma R_{r_\gamma} h_1 = h' = \lim_\gamma R_{r_\gamma} h_2$, it follows necessarily that $h_1 = h_2$.

We denote the classes of minimal, point distal and distal functions on G by $MIN(G)$, $PD(G)$ and $D(G)$, respectively. Clearly distal functions are point distal, and point distal functions are minimal [7, 11, 1]. The functions in Examples 3, (i) and (ii), are distal, but the one in (iii) is in $MIN(G) \setminus PD(G)$. (A function f that is in $PD(\mathbf{Z}) \setminus (D(\mathbf{Z}) \cup BAP(\mathbf{Z}))$ is defined by $f(n) = \cos n / |\cos n|$.)

We quote two theorems.

Theorem 4 [7,11]. Let $f \in \mathscr{U}_r(G)$. Then $f \in MIN(G)$ if and only if:

$$(*) \qquad \begin{cases} \text{for all } \epsilon > 0 \text{ and finite } F \subset G, \text{ there is a finite} \\ K_{\epsilon,F} \subset G \text{ such that} \\ K_{\epsilon,F}\{s \in G \mid |f(ts) - f(t)| < \epsilon \text{ for all } t \in F\} = G. \end{cases}$$

Theorem 5 [14]. $BAP(G) \subset MIN(G)$.

The condition (*) in Theorem 4 looks similar to the (1') formulation of the definition of Bohr almost periodicity; indeed, one can show directly that a Bohr almost periodic function satisfies (*). The proof of Theorem 5 given in [14] shows that, if $f \in BAP(G)$, $h \in X_f$ and $\epsilon > 0$, then there is a $t \in G$ such that

$$(**) \qquad \|R_t h - f\| \leq \epsilon$$

(which proves $f \in MIN(G)$).

Remarks 6. (i) (**) is equivalent to $\| h - R_{t^{-1}} f \| \leq \epsilon$, from which we conclude that, for an $f \in BAP(G)$, X_f, which is the pointwise closure of $R_G f$, equals the norm closure of $R_G f$; we write

$$\overline{R_G f}^{\,p} = \overline{R_G f}^{\,\|\,\|}. \tag{1}$$

It follows from 2(b) that $X_f \subset BAP(G)$.

(ii) Clearly an $f \in \mathscr{U}_r(G)$ that satisfies (1) is in $MIN(G)$. However, not all minimal functions f satisfy (1). A class of minimal functions that do not satisfy (1) is $PD(G) \setminus D(G)$, hence $BAP \cap PD = BAP \cap D$.

(iii) Suppose an $f \in \mathscr{U}_r$ satisfies (1). Does this always imply $f \in BAP$? Not without some connectivity hypothesis. For, suppose $f \in BAP \setminus AP$ on some group G. Then some of the K_ϵ's in Definition 1 cannot be chosen finite, hence f is not Bohr almost periodic on the discrete group G_d. But (1) still holds for f on G_d.

In Example 3 (iii) we pointed out that $BAP(G)$ need not form a linear space. Here are two more unusual aspects of $BAP(G)$.

(a) If G satisfies $BAP \setminus AP \neq \emptyset$, consider the identity map $\imath : G_d \to G$. Although $\imath$ is a continuous homomorphism, the adjoint map $\imath^*$, $\imath^*(f) := f \circ \imath$, does not map $BAP(G)$ into $BAP(G_d)$. (Of course, $\imath^* (AP(G)) \subset AP(G_d)$, etc.)

(b) Let G and f be as in Example 3 (iii). Then *every* $\mathbf{T}$-valued function h on the subgroup $\{1\} \times \mathbf{T}_d$ extends to a function $R_{(h,1)} f \in BAP(G)$. (Note that, if H_1 is a subgroup of a group H and $f \in AP(H)$, for example, then the restriction of f to H_1 is in $AP(H_1)$.)

We quote two more theorems.

THEOREM 7 [14]. Let ψ be a continuous open homomorphism of G_1 onto G_2. Then $\psi^*(BAP(G_2)) \subset BAP(G_1)$.

THEOREM 8 (A. L. T. Paterson). Let G be an amenable locally compact group. Then each $f \in BAP(G)$ is left almost convergent.

We refer the reader to [9, 15, 16] for amenability. A function $f \in \mathscr{U}_r(G)$ is *left almost convergent* if the set

$$\{\mu(f) \mid \mu \text{ is left invariant mean on } \mathscr{U}_r(G)\}$$

is a singleton. Paterson proved Theorem 8 by showing that an $f \in BAP(G)$ has a constant function in its norm closed convex hull.

In conclusion we remark that one can consider Bohr almost periodic functions on topological groups that are not locally compact. All the results here go through unchanged in this more general setting. One can even extend the setting to semi-topological groups; in this setting an $f \in BAP(G)$ is defined to satisfy the condition of Definition 1 and also to be in $\mathscr{U}_r(G)$.

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